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A Characterization of Compactly Supported Both *m* and *n* Refinable Distributions*

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In this paper, we give a characterization of compactly supported distributions which are both m and n refinable for some integer pair (m, n). © 1999 Academic Press
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1. INTRODUCTION

Define the Fourier transform of an integrable function f by

$$\hat{f}(\xi)=\int_{\mathbb{R}} e^{-ix\xi} f(x)\,dx$$

and the one of a compactly supported distribution by usual interpretation. For any integer $m\geqslant 2$, a compactly supported distribution ϕ is said to be *m refinable* if ϕ satisfies the *refinement equation*

$$\phi=\sum_{j\in\mathbb{Z}} c_j\phi(m\cdot -j) \tag{1.1}$$

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and $\hat{\phi}(0) = 1$, where the sequence $\{c_j\}_{j \in \mathbb{Z}}$ satisfies $\sum_{j \in \mathbb{Z}} c_j = m$ and $c_j \neq 0$ for all but finitely many $j \in \mathbb{Z}$. In this paper, a refinable distribution means a compactly supported distribution which is m refinable for some $m \geqslant 2$. Refinable distribution arises in many contexts, such as subdivision scheme and construction of various wavelets (see for instance [1, 2, 5]). Typical examples of refinable distributions are B-splines and Daubechies' scaling functions.

Define the m *symbol* of the refinable distribution ϕ in (1.1) by

$$H_m(z) = \frac{1}{m} \sum_{j \in \mathbb{Z}} c_j z^j.$$

By taking the Fourier transform at each side of (1.1), we obtain

$$\hat{\phi}(\xi) = H_m(e^{-i\xi/m})\, \hat{\phi}(\xi/m). \tag{1.2}$$

From (1.2), we see that an m refinable distribution must be m^r refinable for all integers $r \geqslant 1$. Furthermore its corresponding m^r symbol is $\prod_{j=0}^{r-1} H_m(z^{m^j})$, where H_m is its m symbol. This motivates us to consider the converse—whether a distribution which is m^r refinable for all $r \geqslant 2$ is necessarily m refinable. In this paper, we discuss the following question relating to an even stronger statement.

Problem 1. Let r and s be two relatively prime integers. Is it true that a distribution which is both m^r and m^s refinable is necessarily m refinable?

A compactly supported distribution is said to be *totally refinable* if it is m refinable for all $m \geqslant 2$. Define *B-spline* B_k, $k \geqslant 0$ by

$$\hat{B}_k(\xi) = \left(\frac{1 - e^{-i\xi}}{i\xi}\right)^k.$$

Then the B_k, $k \geqslant 0$ are totally refinable. It motivates us to consider the converse—whether B-splines are the only totally refinable distributions. In this paper, we discuss the following question relating to an even stronger statement.

Problem 2. For which class of integer pairs (m, n) is a compactly supported distribution that is both m and n refinable necessarily essentially a B-spline?

Recall that a compactly supported p refinable distribution is p^r refinable. Then a compactly supported distribution, which is both m and n refinable, need not to be a B-spline if the integer pair (m, n) is (p^r, p^s) for some integers $r, s \geqslant 1$ and $p \geqslant 2$.

Problem 2 is of interest by itself. In [3], Cohen *et al.* proved that the smoothness and approximation order go hand-in-hand for a totally refinable space. The reader refer [3] to the definition of totally refinable spaces. In fact, the space spanned by the integer translates of a totally refinable function is an important class of totally refinable spaces. So to study Problem 2 is helpful to understand the totally refinable spaces. In recent years, some authors have tried to understand when a refinable distribution is essentially a B-spline. Lawton *et al.* proved in [6] that a refinable piecewise polynomial is essentially a finite linear combination of integer translates of a B-spline. In [9], the first named author showed that a compactly supported distribution, which is piecewise smooth and m refinable for some $m \geqslant 2$, is essentially a B-spline.

In this paper, we give an affirmative answer to Problem 1 under some minor restrictions on the refinable distribution and identify certain classes of integer pairs (m, n) for the solution to Problem 2.

To state our results, we fix some terminologies. A compactly supported distribution ϕ is said to be *linearly independent to its integer translates*, or *linearly independent* for short, if

$$\sum_{j \in \mathbb{Z}} d_j \phi(\cdot - j) \equiv 0 \qquad \text{on } \mathbb{R} \text{ implies} \quad d_j = 0, \quad \forall j \in \mathbb{Z}.$$

We say that an integer pair (m, n) is of *type I* if there exist integers $r, s \geqslant 1$ and $p \geqslant 2$ such that $m = p^r$ and $n = p^s$. For $l \geqslant 2$, an integer pair (m, n) is said to be of *type l* if it is not of type $l-1$ and there exist integers $r_i, s_i \geqslant 0$ and $p_i \geqslant 2$, $i = 1, 2, ..., l$ such that p_i, $1 \leqslant i \leqslant l$ are pairwise relatively prime, $m = \prod_{i=1}^{l} p_i^{r_i}$ and $n = \prod_{i=1}^{l} p_i^{s_i}$. For example (9, 27) is of type I, (12, 18) is of type II and $(2^2 \cdot 3 \cdot 5, 3^2 \cdot 5) = (300, 45)$ is of type III. In this paper, we prove the results that only involve integer pairs of type I, II, and III.

THEOREM 1. *Let r and s be two relatively prime integers, and let $m \geqslant 2$ be an integer. Assume that the compactly supported distribution ϕ is linearly independent. Then ϕ is both m^r and m^s refinable if and only if it is m refinable.*

The condition for the linear independence of ϕ in Theorem 1 cannot be left out. For example, the distribution ϕ defined by

$$\hat{\phi}(\xi) = \frac{e^{i\xi} - 1}{i\xi} \times \frac{e^{2i\xi} - 2\cos(2\pi/m^2)\, e^{i\xi} + 1}{2 - 2\cos(2\pi/m^2)}$$

is m^r refinable for all $r \geqslant 2$, but not m refinable.

THEOREM 2. *Let (m, n) be an integer pair of type II or of type III. Assume that the compactly supported distribution ϕ is linearly independent. Then ϕ is both m and n refinable if and only if there exist a B-spline B_k and an integer s such that $s(n-1)/(m-1)$ is still an integer and $\phi = B_k(\cdot - s/(m-1))$.*

We say that a Laurent polynomial P is *m closed* if $P(z^m)/P(z)$ is still a Laurent polynomial. If the condition for the linear independence of ϕ in Theorem 2 is left out, then we have

THEOREM 3. *Let (m, n) be an integer pair of type II or of type III. Then ϕ is both m and n refinable if and only if there exists an integer s such that $s(n-1)/(m-1)$ is an integer, and a B-spline B_k and a sequence $\{d_j\}_{j\in\mathbb{Z}}$ with finite length such that $(1-z)^k \sum_{j\in\mathbb{Z}} d_j z^j$ is both m and n closed, and*

$$\phi = \sum_{j\in\mathbb{Z}} d_j B_k\left(\cdot - \frac{s}{m-1} - j\right).$$

From Theorem 3, it follows that a totally refinable distribution is a finite linear combination of integer translates of a B-spline. So we believe that the following assertion is true.

Conjecture. Let the integer pair (m, n) be not of type I. If a compactly supported distribution is both m and n refinable, then it is essentially a finite combination of the integer translates of a B-spline.

Let us briefly describe the ideas to prove our theorems. The proofs of one direction follow from the facts that a B-spline is m refinable for all $m \geqslant 2$ and that an m refinable distribution is m^r refinable for all integer $r \geqslant 1$. To give the proofs of another direction, we need two basic assertions. The first one says that both m and n refinability of the distribution ϕ is equivalent to

$$H_m(z^n)\ H_n(z) = H_n(z^m)\ H_m(z)$$

on the corresponding m and n symbols H_m and H_n (see Lemma 1 for precise statement). The second one says that a compactly supported distribution, which is both m and n refinable, is also m/n refinable if it is linearly independent and $m/n \geqslant 2$ is still an integer (see Lemma 2 for precise statement). Then we may use Lemma 2 to prove Theorem 1.

The first step to prove Theorem 2 is to simplify integer pairs in Theorem 2 by Lemma 2. In fact it suffices to consider integer pairs (m, n) with m and n being relatively prime, or satisfying $m = pd$ and $n = qd$ for some pairwise relatively prime integers p, q and d. The key step is to prove that the corresponding m symbol H_m can be written as

$$H_m(z) = \left(\frac{1-z^m}{m-mz}\right)^k \frac{P(z^m)}{P(z)}$$

for some Laurent polynomial P with $P(1)=1$ (see Lemmas 3 and 4 for precise statement). At last we show that the Laurent polynomial P above equals z^s for some integer s.

In order to prove Theorem 3, by Theorem 2 we only need to show that for a both m and n refinable distribution ϕ, there exist a compactly supported distribution ϕ_1 and a sequence $\{d_j\}_{j\in\mathbb{Z}}$ with finite length such that ϕ_1 is linearly independent, both m and n refinable, and $\phi = \sum_{j\in\mathbb{Z}} d_j \phi_1(\cdot - j)$ (see Lemma 7 for precise statement).

The paper is organized as follows. In Section 2, we give some basic assertions and the proof of Theorem 1. Section 3 contains the proof of Theorem 2. Theorem 3 is proved in Section 4.

2. PROOFS OF THEOREM 1

To prove our theorems, we need some lemmas.

LEMMA 1. *Let m and $n \geqslant 2$ be two integers. If a compactly supported distribution ϕ is both m and n refinable, then the corresponding m symbol H_m and n symbol H_n satisfy*

$$H_m(z^n)\, H_n(z) = H_n(z^m)\, H_m(z). \tag{2.1}$$

Conversely if Laurent polynomials H_m and H_n satisfy (2.1) and $H_m(1) = H_n(1) = 1$, then there exists a compactly supported distribution ϕ such that it is both m and n refinable, and H_m and H_n are the corresponding m and n symbols respectively.

Proof. Let ϕ be both m and n refinable. Then it follows from (1.2) that

$$\hat{\phi}(\xi) = H_m(e^{-i\xi/m})\, \hat{\phi}\left(\frac{\xi}{m}\right) = H_m(e^{-i\xi/m})\, H_n(e^{-i\xi/(mn)})\, \hat{\phi}\left(\frac{\xi}{mn}\right)$$

and

$$\hat{\phi}(\xi) = H_n(e^{-i\xi/n})\, \hat{\phi}\left(\frac{\xi}{n}\right) = H_n(e^{-i\xi/n})\, H_m(e^{-i\xi/(mn)})\, \hat{\phi}\left(\frac{\xi}{mn}\right).$$

Recall that $\hat{\phi}$ is a nonzero analytic function. Then

$$H_m(e^{-in\xi})\, H_n(e^{-i\xi}) = H_n(e^{-im\xi})\, H_m(e^{-i\xi})$$

and (2.1) follows.

Let H_m and H_n satisfy (2.1) and $H_m(1) = H_n(1) = 1$. Define

$$\Phi(\xi) = \prod_{j=1}^{\infty} H_m(e^{-i\xi/m^j}). \tag{2.2}$$

Then $\Phi(0) = 1$. It is easy to show that the right hand side of (2.2) converges uniformly on any compact set of the complex plane $\mathbb{C}$. Hence $\Phi(\xi)$ is an analytic function. Furthermore there exists a constant C such that $|\Phi(\xi)| \leqslant C(1+|\xi|)^C e^{C\,|\mathrm{Im}\,\xi|}$, where $\mathrm{Im}\,\xi$ denotes the imaginary part of a complex number ξ. Thus there exists a compactly supported distribution ϕ by the Paley–Wiener theorem such that $\Phi = \hat{\phi}$. Hence it remains to prove that ϕ is both m and n refinable. Obviously ϕ is m refinable by (2.2). To prove n refinability of ϕ, we introduce an auxiliary function

$$g(\xi) = \hat{\phi}(n\xi)/\hat{\phi}(\xi) = \Phi(n\xi)/\Phi(\xi).$$

Obviously g is continuous at the origin and $g(0) = 1$. By (2.1) and (2.2), we get

$$g(\xi) = \frac{H_m(e^{-in\xi/m})\, \hat{\phi}(n\xi/m)}{H_m(e^{-i\xi/m})\, \hat{\phi}(\xi/m)} = \frac{H_n(e^{-i\xi})}{H_n(e^{-i\xi/m})}\, g\left(\frac{\xi}{m}\right).$$

Hence

$$g(\xi) = \frac{H_n(e^{-i\xi})}{H_n(e^{-i\xi/m^k})}\, g\left(\frac{\xi}{m^k}\right)$$

for all $k \geqslant 1$ and $g(\xi) = H_n(e^{-i\xi})$ by letting k tend to infinity. This shows that ϕ is n refinable. By the procedure above, we see that H_m and H_n are the m and n symbols of the refinable distribution ϕ respectively. ∎

For $z_0 \in \mathbb{C}\backslash\{0\}$, we say that a Laurent polynomial P has *m symmetric roots* z_0 if $P(z_0 \omega_m^s) = 0$ for all $0 \leqslant s \leqslant m-1$, where $\omega_m = e^{2\pi i/m}$ is the mth root of unity. A Laurent polynomial P is said to *have no m symmetric roots* if all $z_0 \in \mathbb{C}\backslash\{0\}$ are not m symmetric roots of P.

LEMMA 2. *Let m and n be two integers such that $m/n \geqslant 2$ is still an integer. If ϕ is linearly independent, and both m and n refinable, then ϕ is m/n refinable.*

Proof. Let H_m and H_n be the m and n symbol of the refinable distribution ϕ respectively. Then H_n has no n symmetric roots and H_m has no m symmetric roots by the linear independence of ϕ. By Lemma 1, we have

$$H_n(z)\, H_m(z^n) = H_m(z)\, H_n(z^m). \tag{2.3}$$

Write

$$H_m(z) = H_{1,m}(z)\, H_{2,m}(z^n)$$

such that $H_{1,m}$ has no n symmetric roots and $H_{1,m}(1) = 1$. Then all n symmetric roots of the left hand side of (2.3) are those of $H_m(z^n)$ and all n symmetric roots of the right hand side of (2.3) are those of $H_{2,m}(z^n)\, H_n(z^m)$. Therefore by (2.3) we get

$$H_m(z) = H_{2,m}(z)\, H_n(z^{m/n})$$

and

$$H_{1,m}(z) = H_n(z).$$

Replacing H_n and H_m in (2.3) by the formulas above, we obtain

$$H_n(z)\, H_{2,m}(z^n) = H_{2,m}(z)\, H_n(z^{m/n}).$$

Hence Lemma 2 follows from Lemma 1 and the above formula of H_n and $H_{2,m}$. ∎

Proof of Theorem 1. Obviously it suffices to prove that ϕ is m refinable when ϕ is m^r and m^s refinable. If r or s equals 1, then the assertion follows. Inductively we assume that the assertion holds for all relatively prime integers $r \leqslant k$ and $s \leqslant k$. Now we prove the assertion when $r \leqslant k+1$ and $s \leqslant k+1$ are relatively prime. Without loss of generality we assume $r > s$. Set $r' = r - s$. Then $r' \leqslant k$, $s \leqslant k$, and r' and s are also relatively prime. Furthermore ϕ is $m^{r'} = m^r/m^s$ refinable by Lemma 2. Thus ϕ is m refinable by the inductive assumption. Hence the assertion holds when $r \leqslant k+1$ and $s \leqslant k+1$ are relatively prime. ∎

3. PROOF OF THEOREM 2

A Laurent polynomial $P(z)$ is said to be a *normalized polynomial* if $P(z)$ is a polynomial and satisfies $P(0) \neq 0$ and $P(1) = 1$. Denote the set of all nonzero roots of a Laurent polynomial P, taking multiplicities into account, by $Z(P)$. If z_0 is a root of multiplicity m, we may distinguish its repeated occurrence in some way, such as $z_0 \times 1$, $z_0 \times 2, ..., z_0 \times m$. For example,

$$Z(P) = \{i \times 1, i \times 2, -i \times 1, -i \times 2\}$$

when $P(z) = z(z^2+1)^2$. But we abandon such vigor and write simply

$$Z(P) = \{i, i, -i, -i\}.$$

Then the cardinality of the above set of roots of the polynomial $z(z^2+1)^2$ is 4. For any natural number r, let $Z(P)^r$ be the set of all z_0^r with $z_0 \in Z(P)$ and $Z(P) \times Z(Q)$ be the set of all $z_0 u_0$ with $z_0 \in Z(P)$ and $u_0 \in Z(Q)$. For the above example, $Z(P)^2 = \{-1, -1, -1, -1\}$ and $Z(P) \times \{-1, 1\} = \{i, i, i, i, -i, -i, -i, -i\}$.

LEMMA 3. *Let m and n be relatively prime integers. If H_m has no m symmetric roots, H_n has no n symmetric roots, and H_m and H_n satisfy*

$$H_m(z)\, H_n(z^m) = H_n(z)\, H_m(z^n),$$

then there exist a normalized polynomial P and an integer $k \geqslant 0$ such that P is m and n closed, and

$$H_m(z) = \left(\frac{1-z^m}{m-mz}\right)^k \frac{P(z^m)}{P(z)}, \qquad H_n(z) = \left(\frac{1-z^n}{n-nz}\right)^k \frac{P(z^n)}{P(z)}.$$

Proof. Let $A(z)$ be the maximal common factor of $H_m(z)$ and $H_n(z)$ with $A(1) = 1$. Then

$$Q(z) = \frac{A(z)\, H_n(z^m)}{H_n(z)} = \frac{A(z)\, H_m(z^n)}{H_m(z)}$$

is a polynomial by the assumption on H_m and H_n. Furthermore we have

Claim 1. $Q(z)$ has no m symmetric roots.

On the contrary, there exists $z_0 \in \mathbb{C}$ such that $Q(z_0 \omega_m^s) = 0$ for all $0 \leqslant s \leqslant m-1$. Observe that $\{\omega_m^s; 0 \leqslant s \leqslant m-1\} = \{\omega_m^{sn}; 0 \leqslant s \leqslant m-1\}$ when m and n are relatively prime. Then $H_m(z_0^n \omega_m^s) = 0$ for all $0 \leqslant s \leqslant m-1$, which contradicts to the assumption on H_m.

Similarly by the assumption on H_n we have

Claim 2. $Q(z)$ has no n symmetric roots.

Thus it follows from Claims 1 and 2 that $A(z) = 1$ and

$$Z(H_n)^m = Z(H_n), \qquad Z(H_m)^n = Z(H_m). \tag{3.1}$$

Write

$$H_n(z) = C \prod_{z_0 \in Z(H_n)} (z - z_0).$$

Then $H_n(z) = C \prod_{z_0 \in Z(H_n)} (z - z_0^m)$ by (3.1) and

$$Q(z) = \prod_{z_0 \in Z(H_n)} \frac{z^m - z_0^m}{z - z_0} = \prod_{z_0 \in Z(H_n)} \prod_{s=1}^{m-1} (z - z_0 \omega_m^s).$$

Similarly we have

$$Q(z) = \prod_{u_0 \in Z(H_m)} \prod_{t=1}^{n-1} (z - u_0 \omega_n^t).$$

Hence we get

$$Z(Q) = Z(H_n) \times \{\omega_m^s;\, 1 \leqslant s \leqslant m-1\} = Z(H_m) \times \{\omega_n^t;\, 1 \leqslant t \leqslant n-1\}. \quad (3.2)$$

By (3.1) and (3.2), we obtain

$$Z(H_m) \times \{1, 1, ..., 1\}_{n-1} = Z(H_n)^n \times \{\omega_m^s;\, 1 \leqslant s \leqslant m-1\} \qquad (3.3)$$

and

$$Z(H_n) \times \{1, 1, ..., 1\}_{m-1} = Z(H_m)^m \times \{\omega_n^t;\, 1 \leqslant t \leqslant n-1\} \qquad (3.4)$$

where $\{\zeta_0, \zeta_0, ..., \zeta_0\}_k$ is the set of all roots of $(z - \zeta_0)^k$ for $\zeta_0 \in \mathbb{C} \backslash \{0\}$. Thus we have

Claim 3. There exists a polynomial P_1 such that $Z(H_n)^n = Z(P_1) \times \{1, 1, ..., 1\}_{n-1}$.

On the contrary, there exist $z_1, z_2 \in Z(H_n)^n$ and $1 \leqslant s_1 \leqslant n-1$ such that $z_1 = z_2 \omega_m^{s_1}$ by (3.3). Hence

$$\{z_1 \omega_m^s;\, 0 \leqslant s \leqslant m-1\} \subset Z(H_n)^n \times \{\omega_m^s;\, 1 \leqslant s \leqslant m-1\}$$

and H_m has m symmetric root z_1 by (3.3), which contradicts the assumption on H_m.

Combining (3.1), (3.3), and Claim 3, we obtain

$$Z(H_m) = Z(P_1) \times \{\omega_m^s;\, 1 \leqslant s \leqslant m-1\} \qquad (3.5)$$

and

$$Z(P_1)^n \times \{\omega_m^s;\, 1 \leqslant s \leqslant m-1\} = Z(P_1) \times \{\omega_m^s;\, 1 \leqslant s \leqslant m-1\}.$$

Furthermore we have

Claim 4. $Z(P_1)=Z(P_1)^n$.

On the contrary, there exist $z_1\in Z(P_1)$, $z_2\in Z(P_1)^n$ and $1\leqslant s_1\leqslant m-1$ such that $z_1=z_2\omega_m^{s_1}$. Hence H_m has m symmetric roots z_1 by (3.1) and (3.5), which contradicts the assumption on H_m.

Similarly by (3.1), (3.2), (3.4), and the assumption on H_n there exists a polynomial P_2 such that

$$\begin{cases} Z(H_n)=Z(P_2)\times\{\omega_n^t;\,1\leqslant t\leqslant n-1\} \\ Z(P_2)=Z(P_2)^m. \end{cases} \tag{3.6}$$

By (3.2), (3.5), and (3.6), we obtain

$$\begin{aligned} &Z(P_1)\times\{\omega_n^t;\,1\leqslant t\leqslant n-1\}\times\{\omega_m^s;\,1\leqslant s\leqslant m-1\} \\ &\quad =Z(P_2)\times\{\omega_n^t;\,1\leqslant t\leqslant n-1\}\times\{\omega_m^s;\,1\leqslant s\leqslant m-1\}. \end{aligned}$$

Furthermore we have

Claim 5. $Z(P_1)=Z(P_2)$.

On the contrary, there exist $z_1\in Z(P_1)$, $z_2\in Z(P_2)$, $0\leqslant s_1\leqslant m-1$ and $0\leqslant t_1\leqslant n-1$ such that $(s_1,t_1)\neq(0,0)$ and $z_1=z_2\omega_m^{s_1}\omega_n^{t_1}$. From (3.2), (3.5), and (3.6), it follows that

$$Q(z_1\omega_m^s\omega_n^t)=0, \qquad \forall 1\leqslant s\leqslant m-1,\quad 0\leqslant t\leqslant n-1$$

when $s_1=0$,

$$Q(z_1\omega_m^s\omega_n^t)=0, \qquad \forall 0\leqslant s\leqslant m-1,\quad 1\leqslant t\leqslant n-1$$

when $t_1=0$ and

$$Q(z_1\omega_m^s\omega_n^t)=0, \qquad \forall 0\leqslant s\leqslant m-1,\quad 0\leqslant t\leqslant n-1$$

when $s_1\neq 0$ and $t_1\neq 0$. Hence Q has m or n symmetric roots, which contradicts Claims 1 and 2.

Write $P_1(z)=C(1-z)^k\,P_0(z)$ with $P_0(1)=1$. Hence Lemma 3 follows by (3.5), (3.6), Claims 4 and 5, and letting $P=P_0$. ∎

LEMMA 4. *Let $p, q, d\geqslant 2$ be pairwise relatively prime integers, $m=pd$ and $n=qd$. Assume that the normalized polynomials H_m and H_n have no m and n symmetric roots respectively. If H_n and H_n satisfy* (2.1), *then there* *exist a normalized polynomial P and an integer $k \geqslant 0$ such that P is m and n closed, and*

$$H_m(z) = \left(\frac{1-z^m}{m-mz}\right)^k \frac{P(z^m)}{P(z)}, \qquad H_n(z) = \left(\frac{1-z^n}{n-nz}\right)^k \frac{P(z^n)}{P(z)}.$$

Obviously Lemma 4 follows from Lemmas 5 and 6 below.

LEMMA 5. *Let m, n, p, q, d, H_m, H_n be as in Lemma 4. If H_m* and *H_n satisfy* (2.1), *then*

$$\begin{cases} H_m(z) = H_{m,1}(z^d)\, B(z) = H_{m,2}(z)\, C(z^p) \\ H_n(z) = H_{n,1}(z^d)\, B(z) = H_{n,2}(z)\, C(z^q), \end{cases} \tag{3.7}$$

where $B(z)$, $C(z)$, and $H_{n,i}(z)$, $H_{m,i}(z)$, $i=1,2$ are normalized polynomials. Furthermore $B(z)$ and $C(z)$ have no d symmetric roots, $H_{m,i}(z)$, $i=1,2$ has no p symmetric roots and $H_{n,i}(z)$, $i=1,2$ has no q symmetric roots.

Proof. Write

$$H_m(z) = H_{m,1}(z^d)\, B_1(z) = H_{m,2}(z)\, C_1(z^p),$$

$$H_n(z) = H_{n,1}(z^d)\, B_2(z) = H_{n,2}(z)\, C_2(z^q),$$

such that $H_{n,i}(z), H_{m,i}(z), B_i(z), C_i(z)$, $i=1,2$ are normalized polynomials, and $B_i(z)$, $i=1,2$ has no d symmetric roots, $H_{m,2}(z)$ has no p symmetric roots, and $H_{n,2}(z)$ has no q symmetric roots. By the assumptions on H_m and H_n we see that $C_i(z)$, $i=1,2$ has no d symmetric roots, $H_{m,1}(z)$ has no p symmetric roots and $H_{n,1}(z)$ has no q symmetric roots. Thus it suffices to prove that $B_1(z) = B_2(z)$ and $C_1(z) = C_2(z)$.

We first show that $B_1(z) = B_2(z)$. By (2.1), we have

$$B_1(z)\, H_{m,1}(z^d)\, H_n(z^{dp}) = B_2(z)\, H_{n,1}(z^d)\, H_m(z^{dq}). \tag{3.8}$$

It is easy to see that all d symmetric roots of the left hand side of (3.8) are those of $H_{m,1}(z^d)\, H_n(z^{dp})$, and all d symmetric roots of the right hand side of (3.8) are those of $H_{n,1}(z^d)\, H_m(z^{dq})$. Thus we have $Z(B_1) = Z(B_2)$. Hence from $B_1(0) \neq 0$, $B_2(0) \neq 0$, and $B_1(1) = B_2(1)$, it follows that

$$B_1(z) = B_2(z).$$

Next we prove that $C_1(z) = C_2(z)$. Obviously (2.1) can be written as

$$H_m(z)\, H_{n,2}(z^{dp})\, C_2(z^{dpq}) = H_n(z)\, H_{m,2}(z^{dq})\, C_1(z^{dpq}). \tag{3.9}$$

Hence we have

Claim 6. All dpq symmetric roots of the left hand side of (3.9) are those of $C_2(z^{dpq})$.

On the contrary, there exists a complex number z_0 such that

$$H_m(z_0\omega_{dpq}^u)\, H_{n,2}(z_0^{dp}\omega_q^u)=0, \qquad \forall 0\leqslant u\leqslant dpq-1.$$

Hence

$$H_m(z_0\omega_{dpq}^{s+tq})\, H_{n,2}(z_0^{dp}\omega_q^s)=0, \qquad \forall 0\leqslant s\leqslant q-1, \quad 0\leqslant t\leqslant dp-1. \quad (3.10)$$

Recall that $H_{n,2}(z)$ has no q symmetric roots. Therefore there exists $0\leqslant s_0\leqslant q-1$ such that $H_{n,2}(z_0^{dp}\omega_q^{s_0})\neq 0$. Hence $H_m(z_0\omega_{spq}^{s_0}\omega_m^t)=0$ for all $0\leqslant t\leqslant m-1$ by (3.10), which contradicts to the assumption on H_m.

Similarly we have

Claim 7. All dpq symmetric roots of the right hand side of (3.9) are those of $C_1(z^{dpq})$.

Therefore by Claims 6 and 7 we have $Z(C_1)=Z(C_2)$. Recall that $C_i(z)$, $i=1,2$ are normalized polynomials. Then

$$C_1(z)=C_2(z).$$

Hence Lemma 5 follows by letting $B(z)=B_1(z)$ and $C(z)=C_1(z)$. ∎

LEMMA 6. *Let m, n, p, q, d and $H_m(z)$, $H_n(z)$, $B(z)$, $C(z)$, $H_{n,i}(z)$, $H_{m,i}(z)$, $i=1,2$ be as in Lemma 5. Then there exist normalized polynomials $P_i(z)$, $i=0,1,2$ and an integer $k\geqslant 0$ such that*

$$\begin{cases} H_{m,1}(z)=(1-z^p)^k/(p-pz)^k\times P_1(z^p)/P_0(z), \\ H_{m,2}(z)=(1-z^p)^k/(p-pz)^k\times P_2(z^p)/P_1(z), \\ H_{n,1}(z)=(1-z^q)^k/(q-qz)^k\times P_1(z^q)/P_0(z), \\ H_{n,2}(z)=(1-z^q)^k/(q-qz)^k\times P_2(z^q)/P_1(z), \\ B(z)=(1-z^d)^k/(d-dz)^k\times P_0(z^d)/P_1(z), \\ C(z)=(1-z^d)^k/(d-dz)^k\times P_1(z^d)/P_2(z), \end{cases} \quad (3.11)$$

and $P_0(z^d)/P_1(z)$, $P_1(z^d)/P_2(z)$, $P_1(z^p)/P_0(z)$, $P_1(z^q)/P_0(z)$, $P_2(z^p)/P_1(z)$ and $P_2(z^q)/P_1(z)$ are normalized polynomials.

Proof. By (3.7) and (3.8), we obtain

$$\begin{aligned} H_{m,1}(z^d)\, B(z)&=H_{m,2}(z)\, C(z^p), \\ H_{n,1}(z^d)\, B(z)&=H_{n,2}(z)\, C(z^q), \\ H_{m,1}(z)\, H_{n,2}(z^p)&=H_{n,1}(z)\, H_{m,2}(z^q). \end{aligned} \quad (3.12)$$

First we prove that

$$\begin{aligned} Z(H_{m,2}) &= Z(H_{m,1})^q, \\ Z(H_{m,1}) &= Z(H_{m,2})^d, \\ Z(H_{m,1}) &= Z(H_{m,1})^n, \end{aligned} \tag{3.13}$$

and

$$\begin{aligned} Z(H_{n,2}) &= Z(H_{n,1})^p, \\ Z(H_{n,1}) &= Z(H_{n,2})^d, \\ Z(H_{n,1}) &= Z(H_{n,1})^m. \end{aligned} \tag{3.14}$$

Since we can prove (3.14) by almost the same argument as the one of (3.13), we only give the detail of the proof of (3.13) here. Let $R_3(z)$ be the maximal common factor between $H_{m,1}(z)$ and $H_{n,1}(z)$ with $R_3(1) = 1$. Set

$$Q_1(z) = \frac{H_{m,2}(z^q)\, R_3(z)}{H_{m,1}(z)}. \tag{3.15}$$

Then $Q_1(z)$ is a normalized polynomial and

$$Q_1(z) = \frac{H_{n,2}(z^p)\, R_3(z)}{H_{n,1}(z)} \tag{3.16}$$

by (3.12). Furthermore we have

Claim 8. $Q_1(z)$ has no p symmetric roots.

On the contrary, there exists $z_0 \in \mathbb{C}$ such that $Q_1(z_0 \omega_p^s) = 0$ for all $0 \leqslant s \leqslant p-1$. Thus $H_{m,2}(z_0^q \omega_p^{sq}) = 0$ for all $0 \leqslant s \leqslant p-1$ by (3.15). By computation, we have $\{\omega_p^{sq}; 0 \leqslant s \leqslant p-1\} = \{\omega_p^s; 0 \leqslant s \leqslant p-1\}$. Therefore $H_{m,2}(z_0^q \omega_p^s) = 0$ for all $0 \leqslant s \leqslant p-1$, which contradicts the property of $H_{m,2}$.

Similarly by (3.16) and the property of $H_{n,2}$ we have

Claim 9. $Q_1(z)$ has no q symmetric roots.

Thus it follows from (3.15), Claims 8 and 9 that

$$Z(H_{m,2}) \subset Z(H_{m,1}/R_3)^q \subset Z(H_{m,1})^q. \tag{3.17}$$

Let $R_4(z)$ be the maximal common factor between $B(z)$ and $H_{m,2}(z)$ with $R_4(1) = 1$, and let

$$Q_2(z) = \frac{R_4(z)\, H_{m,1}(z^d)}{H_{m,2}(z)}.$$

Then $Q_2(z) = C(z^p)\, R_4(z)/B(z)$ is a polynomial by (3.12) and $Q_2(z)$ has no p and d symmetric roots by the same argument as the one used in the proof of (3.17). Therefore we get

$$Z(H_{m,1}) \subset Z(H_{m,2}/R_4)^d \subset Z(H_{m,2})^d. \tag{3.18}$$

Combining (3.17) and (3.18), we get

$$Z(H_{m,2}) \subset Z(H_{m,2})^n. \tag{3.19}$$

Observe that the sets at both sides of (3.19) have the same cardinality. Then $Z(H_{m,2}) = Z(H_{m,2})^n$, $Z(H_{m,1}) = Z(H_{m,2})^d$ and $R_3(z) = R_4(z) = 1$ by (3.17)–(3.19). Hence (3.13) follows.

By (3.15), (3.16), and $R_3(z) = 1$, we have

$$Q_1(z) = \frac{H_{m,2}(z^q)}{H_{m,1}(z)} = \frac{H_{n.2}(z^p)}{H_{n,1}(z)}. \tag{3.20}$$

By the same argument as the one used in the proof of Lemma 3 it follows from (3.13) and (3.20) that

$$Z(Q_1) = Z(H_{m,1}) \times \{\omega_q^s;\ 1 \leqslant s \leqslant q-1\} = Z(H_{n,1}) \times \{\omega_p^t;\ 1 \leqslant t \leqslant p-1\}. \tag{3.21}$$

Hence by (3.13), (3.14), and (3.21) we obtain

$$\begin{cases} Z(H_{n,1}) \times \{1, 1, ..., 1\}_{p-1} = Z(H_{m,1})^m \times \{\omega_q^s;\ 1 \leqslant s \leqslant q-1\} \\ Z(H_{m,1}) \times \{1, 1, ..., 1\}_{q-1} = Z(H_{n,1})^n \times \{\omega_p^t;\ 1 \leqslant t \leqslant p-1\}. \end{cases} \tag{3.22}$$

Then by the same argument as the one used in the proof of Lemma 3, it follows from (3.13), (3.14), (3.22) and the properties of $H_{m,1}$ and $H_{n,1}$ that there exist polynomials $\tilde{P}_1$ and $\tilde{P}_2$ such that

$$\begin{cases} Z(H_{m,1}) = Z(\tilde{P}_1) \times \{\omega_p^s;\ 1 \leqslant s \leqslant p-1\} \\ Z(H_{n,1}) = Z(\tilde{P}_2) \times \{\omega_q^t;\ 1 \leqslant t \leqslant q-1\} \end{cases} \tag{3.23}$$

and

$$Z(\tilde{P}_1)^n = Z(\tilde{P}_1), \qquad Z(\tilde{P}_2)^m = Z(\tilde{P}_2). \tag{3.24}$$

By (3.21) and (3.23), we have

$$\begin{aligned} &Z(\tilde{P}_1) \times \{\omega_p^t;\ 1 \leqslant t \leqslant p-1\} \times \{\omega_q^s;\ 1 \leqslant s \leqslant q-1\} \\ &\quad = Z(\tilde{P}_2) \times \{\omega_p^t;\ 1 \leqslant t \leqslant p-1\} \times \{\omega_q^s;\ 1 \leqslant s \leqslant q-1\}. \end{aligned}$$

Hence by the same argument as the one used in the proof of Lemma 3 it follows from (3.20), Claims 8 and 9 that

$$Z(\tilde{P}_1)=Z(\tilde{P}_2). \tag{3.25}$$

Write

$$\begin{cases} \prod_{u_\alpha \in Z(\tilde{P}_1)} (z-u_\alpha)=c_1(z-1)^k\, P_0(z), \\ \prod_{u_\alpha \in Z(\tilde{P}_1)} (z-u_\alpha^p)=c_2(z-1)^k\, P_1(z), \\ \prod_{u_\alpha \in Z(\tilde{P}_1)} (z-u_\alpha^q)=c_3(z-1)^k\, P_1^*(z), \\ \prod_{u_\alpha \in Z(\tilde{P}_1)} (z-u_\alpha^{pq})=c_4(z-1)^k\, P_2(z), \end{cases} \tag{3.26}$$

where $k \geqslant 0$ and constants c_i, $1 \leqslant i \leqslant 4$ are chosen such that P_i, $i=0, 1, 2$ and P_1^* are normalized polynomials. Here the same integer k is chosen in (3.26) because $u_\alpha^p \neq 1$, $u_\alpha^q \neq 1$ and $u_\alpha^{pq} \neq 1$ when $u_\alpha \neq 1$ by (3.24) and (3.25). Again by (3.24) and (3.25), we obtain

$$P_1(z)=P_1^*(z). \tag{3.27}$$

Hence it follows from (3.13), (3.14), (3.23), (3.26), and (3.27) that

$$H_{m,1}(z)=\left(\frac{z^p-1}{pz-p}\right)^k \frac{P_1(z^p)}{P_0(z)},$$

$$H_{n,1}(z)=\left(\frac{z^q-1}{qz-q}\right)^k \frac{P_1(z^q)}{P_0(z)},$$

$$H_{m,2}(z)=\left(\frac{z^p-1}{pz-p}\right)^k \frac{P_2(z^p)}{P_1(z)},$$

$$H_{n,2}(z)=\left(\frac{z^q-1}{qz-q}\right)^k \frac{P_2(z^q)}{P_1(z)}.$$

Substituting the above formulas of $H_{m,i}$ and $H_{n,i}$, $i=1, 2$ in the first and second equation of (3.12), we obtain

$$\frac{(1-z^m)^k\, P_1(z^m)}{(p-pz^d)^k\, P_0(z^d)}\, B(z)=\frac{(1-z^p)^k\, P_2(z^p)}{(p-pz)^k\, P_1(z)}\, C(z^p)$$

$$\frac{(1-z^n)^k\, P_1(z^n)}{(q-qz^d)^k\, P_0(z^d)}\, B(z)=\frac{(1-z^q)^k\, P_2(z^q)}{(q-qz)^k\, P_1(z)}\, C(z^q).$$

Hence

$$\frac{(1-z^p)^k P_2(z^p)}{(1-z^m)^k P_1(z^m)} C(z^p) = \frac{(1-z^q)^k P_2(z^q)}{(1-z^n)^k P_1(z^n)} C(z^q).$$

It is easy to prove that a rational polynomial Q satisfying $Q(z^p) = Q(z^q)$ is a constant polynomial. Therefore we have

$$C(z) = \left(\frac{1-z^d}{d-dz}\right)^k \frac{P_1(z^d)}{P_2(z)}.$$

Replacing $C(z)$ in (3.28) by the above formula, we get

$$B(z) = \left(\frac{1-z^d}{d-dz}\right)^k \frac{P_0(z^d)}{P_1(z)}.$$

By the construction of P_i, $i = 0, 1, 2$, these polynomials satisfy the required properties of Lemma 6. ▌

Proof of Theorem 2. Let s be an integer such that $s(n-1)/(m-1)$ is still an integer and let $\phi = B_k(\cdot - s/(m-1))$. Then ϕ is linearly independent and

$$\hat{\phi}(\xi) = e^{-is\xi/(m-1)} \left(\frac{1-e^{-i\xi}}{i\xi}\right)^k.$$

Thus we have

$$\hat{\phi}(\xi) = e^{-is\xi/m} \left(\frac{1-e^{-i\xi}}{m-me^{-i\xi/m}}\right)^k \hat{\phi}\left(\frac{\xi}{m}\right)$$

and

$$\hat{\phi}(\xi) = e^{-is'\xi/n} \left(\frac{1-e^{-i\xi}}{n-ne^{-i\xi/n}}\right)^k \hat{\phi}\left(\frac{\xi}{n}\right),$$

where $s' = s(n-1)/(m-1)$. Hence ϕ is m and n refinable. The necessity follows.

Now we prove the sufficiency when the integer pair (m, n) be of type II. Let p_i, r_i, s_i, $i = 1, 2$ be nonnegative integers such that $p_1 \geqslant 2$ and $p_2 \geqslant 2$ are relatively prime, $m = p_1^{r_1} p_2^{r_2}$ and $n = p_1^{s_1} p_2^{s_2}$. Without loss of generality we assume $r_1 s_2 > r_2 s_1$. Set $m' = n^{r_1}/m^{s_1} = p_2^{r_1 s_2 - r_2 s_1}$ and $n' = m^{s_2}/n^{r_2} = p_1^{r_1 s_2 - r_2 s_1}$. Then m' and n' are relatively prime. By the assumption on ϕ and Lemma 2, ϕ is both m' and n' refinable. From Lemma 1 it follows that the m' and n' symbols $H_{m'}$ and $H_{n'}$ of ϕ satisfy

$$H_{m'}(z)\, H_{n'}(z^{m'}) = H_{n'}(z)\, H_{m'}(z^{n'}). \tag{3.29}$$

Write $H_{m'}(z) = z^s \tilde{H}_{m'}(z)$ and $H_{n'}(z) = z^{s'} \tilde{H}_{n'}(z)$, where $\tilde{H}_{m'}$ and $\tilde{H}_{n'}$ are normalized polynomials. Then $s'(m'-1) = s(n'-1)$, and $\tilde{H}_{m'}$ and $\tilde{H}_{n'}$ satisfy (3.29). Define $\tilde{\phi} = \phi(\cdot - s/(m'-1))$. Then $\tilde{\phi}$ is m' and n' refinable, and its m' and n' symbols are $\tilde{H}_{m'}$ and $\tilde{H}_{n'}$, respectively. By Lemma 3, we get

$$\tilde{H}_{m'}(z) = \left(\frac{1 - z^{m'}}{m' - m'z}\right)^k \frac{P(z^{m'})}{P(z)},$$

where P is a normalized polynomial. Hence

$$\hat{\tilde{\phi}}(\xi) = \left(\frac{1 - e^{-i\xi}}{i\xi}\right)^k P(e^{-i\xi}).$$

Obviously $\tilde{\phi}$ is linearly dependent if the normalized polynomial P above is not a constant. This proves $P(z) = 1$ and $\tilde{\phi} = B_k$. It is obvious that $B_k(\cdot - t)$, $t \in \mathbb{R}$ is m refinable if and only if $(m-1)t \in \mathbb{Z}$. Hence the sufficiency follows when the integer pair (m, n) is of type II.

At last we prove the sufficiency when the integer pair (m, n) is of type III. Let p_i, r_i, s_i, $i = 1, 2, 3$ be nonnegative integers such that $p_1, p_2, p_3 \geqslant 2$ are pairwise relatively prime, $m = p_1^{r_1} p_2^{r_2} p_3^{r_3}$ and $n = p_1^{s_1} p_2^{s_2} p_3^{s_3}$. Without loss of generality we assume that $r_1/s_1 > r_2/s_2 > r_3/s_3$. Then ϕ is $n^{r_1}/m^{s_1} = p_2^{s_2 r_1 - s_1 r_2} p_3^{s_3 r_1 - s_1 s_3}$ and $m^{s_3}/n^{r_3} = p_1^{r_1 s_3 - r_3 s_1} p_2^{r_2 s_3 - r_3 s_2}$ refinable by Lemma 2 and the assumption on ϕ. Hence after appropriately choosing p_i, $i = 1, 2, 3$, we may assume that $s_1 = r_3 = 0$ and $r_1 = s_3 = 1$. For the above integer pair $(m_*, n_*) = (p_1 p_2^{r_2}, p_2^{s_2} p_3)$, set $p = p_1^{s_2}$, $q = p_3^{r_2}$, $d = p_2^{r_2 s_2}$. Then $m_*^{s_2} = pd$, $n_*^{r_2} = qd$ and p, q, d are pairwise relatively prime. Furthermore ϕ is pd and qd refinable by Lemma 2. By the same argument as the one used in the proof for the integer pairs of type II, it follows from Lemma 4 and the linear independence of ϕ that the pd symbol H_{pd} of ϕ may be written as

$$H_{pd}(z) = z^s \left(\frac{1 - z^{pd}}{pd - pdz}\right)^k,$$

for some integers $k \geqslant 0$ and s. Thus $\phi = B_k(\cdot - s/(pd))$. Hence the sufficiency follows when the integer pair (m, n) is of type III. ∎

4. PROOF OF THEOREM 3

To prove Theorem 3, we need the following lemma.

LEMMA 7. *Let $m, n \geqslant 2$ be two integers, and let compactly supported distribution ϕ be both m and n refinable. Then there exist a compactly supported distribution ϕ_1 and a sequence $\{d_j\}_{j\in\mathbb{Z}}$ with finite length such that ϕ_1 is linearly independent, both m and n refinable, and satisfies*

$$\phi = \sum_{j\in\mathbb{Z}} d_j \phi_1(\cdot - j). \tag{4.1}$$

Proof. It is well known (see [7] for instance) that there exist a compactly supported distribution ϕ_1 and a sequence $\{d_j\}_{j\in\mathbb{Z}}$ with finite length such that (4.1) holds and ϕ_1 is linearly independent. Then it suffices to prove that ϕ_1 are both m and n refinable. Set $D(z) = \sum_{j\in\mathbb{Z}} d_j z^j$. Then by taking the Fourier transform at each side of (4.1), we obtain

$$\hat{\phi}(\xi) = D(e^{-i\xi})\, \hat{\phi}_1(\xi).$$

Hence by the m refinability of ϕ and the linear independence of ϕ_1, we have

$$D(e^{-im\xi})\, \hat{\phi}_1(m\xi) = H_m(e^{-i\xi})\, D(e^{-i\xi})\, \hat{\phi}_1(\xi)$$

and $H_m(z)\, D(z)/D(z^m)$ is a Laurent polynomial. This shows that ϕ_1 is m refinable. Similarly we may prove that ϕ_1 is also n refinable. ∎

Proof of Theorem 3. By Lemma 7, there exist a compactly supported distribution ϕ_1 and a sequence $\{d_j\}_{j\in\mathbb{Z}}$ with finite length such that ϕ_1 is both m and n refinable, linearly independent and $\phi = \sum_{j\in\mathbb{Z}} d_j \phi_1(\cdot - j)$. By Theorem 2, there exist integers $k \geqslant 0$ and s such that $s(n-1)/(m-1)$ is still an integer and $\phi_1 = B_k(\cdot - s/(m-1))$. Therefore

$$\phi = \sum_{j\in\mathbb{Z}} d_j B_k\left(\cdot - j - \frac{s}{m-1}\right). \tag{4.2}$$

By taking the Fourier transform at each side of (4.2), we obtain

$$\hat{\phi}(\xi) = e^{-is\xi/(m-1)} \left(\frac{1-e^{-i\xi}}{i\xi}\right)^k \sum_{j\in\mathbb{Z}} d_j e^{-ij\xi}.$$

Thus $(1-z)^k \sum_{j\in\mathbb{Z}} d_j z^j$ is m and n closed by the m and n refinability of ϕ. ∎

Note added in proof. The conjecture in this paper is solved by X. Dai, Q. Sun, and Z. Zhang in "A Characterization of Compactly Supported Both m and n Refinable Distribution, II," forthcoming.

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